

# IS THE CLAUSIUS INEQUALITY A CONSEQUENCE OF THE SECOND LAW?

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**ABSTRACT.** We present an analysis of the foundations of the well known Clausius inequality. It is shown that, in general, the inequality is not a logical consequence of the Kelvin-Planck formulation of the second law of thermodynamics. Some thought experiments demonstrating the violation of the Clausius inequality are considered. The possibility of experimental detection of the violation is pointed out.

*Key words:* Clausius inequality, second law, feedback control, Szilard engine.

## 1. INTRODUCTION

The Clausius inequality is a well known statement of classical thermodynamics. It asserts that the integrated heat absorbed by a system in a cyclic thermodynamic process, divided by the temperature at which that heat is taken, is bounded from above by zero:

$$\oint \frac{\delta Q}{T} \leq 0. \quad (1)$$

The denominator  $T$  in (1) denotes the temperature of the heat bath from which the system takes heat  $\delta Q$ . In the process the system may be in contact with several baths at different temperatures, one at a time.

In the textbooks on thermodynamics the Clausius inequality is usually considered an equivalent of the second law. One of the aims of this paper is to show that the inequality (1) is not a logical consequence of the Kelvin-Planck formulation of the second law of thermodynamics. In other words, it cannot be deduced properly from the second law, if we make no more assumptions than is necessary for the second law itself.

To show this, we consider a thought experiment with a system called a *xenium engine*. This system undergoes a reversible process in which the fraction  $\delta Q/T$  is not an exact differential. Thus, the Clausius inequality is violated. However, there is no contradiction to the second law. There are in fact two xenium engines, and the violation of the inequality (1) is a consequence of the entropy exchange between them. The only unusual property of the process is the absence of any energy transfer or exchange of particles between this two engines.

At this point the author's intention can easily be misunderstood, so it is appropriate to give some explanation. What we are going to prove is that the Clausius inequality (1) cannot be applied wherever the second law can. So, the xenium engine may be unrealistic and peculiar, but it is not the point. The engine definitely obeys the second law, together with a lot of much more exotic systems, real and imaginary. One cannot possibly expect a perpetuum mobile of the second kind

to be devised by the same means. Thus, this example shows that the Clausius inequality and the second law is not the same.

Another thought experiment considered in this paper is the Szilard engine, invented by L. Szilard in 1929 [11]. Due to its microscopic nature it is much less convenient for analysis than the xenium engine, but, in a sense, more realistic. The fact that the Szilard engine violates the Clausius inequality was obvious since its inception, for it performs work in an isothermal cycle. Moreover, the engine converts heat to work, in apparent contradiction to the second law. The explanation of why the engine cannot break the second law is known for decades. The explanation, however, does not rescue the Clausius inequality. This peculiarity is in essence known to experts but, oddly enough, it has never been formulated in terms of the inequality, for the best of author's knowledge.

In the literature, the analysis of the Szilard engine is always performed by means of statistical mechanics; the only exception is the work of Ishioka and Fuchikami [5], where the Clausius entropy is considered. Usually, the authors show that what is going on does not contradict the second law and stop at this point with no intention to go further. The thermodynamic way of thinking is the opposite: to take the second law for granted and to find out what exactly may go on and what may not. This, of course, is a more difficult problem. The present paper is a modest attempt to address it.

Recently, the problem of the origin of the Clausius inequality in statistical mechanics has attracted some attention. The author would like to stress that in the present paper we follow the thermodynamic approach exclusively. A seemingly related question about the validity of the inequality in statistical physics is actually a different problem. The known proofs of the Clausius inequality by methods of statistical mechanics (e.g. [6]) have little in common with the classical argument by Clausius. So, it is by no means clear what a "statistical" analog of the environment independence condition, which plays a key role in the analysis below, may be. (The author's guess is that it can only be formulated for a quantum system). A problem appears to be more difficult in statistical physics than in classical thermodynamics, which is not unusual.

The open question is whether the Clausius inequality may be broken in a real experiment. The inequality is a falsifiable statement, and one can test it in a laboratory. Someone who is going to test the Clausius inequality needs a thermodynamic framework for the analysis. One of the goals of this paper is to provide such a framework. One may compare this to the post-Newtonian formalism which is a tool in tests of general relativity. However, there is some difference. Unlike the post-Newtonian formalism, the proposed formalism is in perfect agreement with all the basic principles of the "mainstream theory", i.e. classical thermodynamics. The reason is simple. Contrary to the common belief, the Clausius inequality is not a logical equivalent of the second law, but a stronger statement. So, its violation, if it is really possible, can be accepted without devastating consequences to the theory. Due to this subtlety in the internal logic of thermodynamics "to test the Clausius inequality" does not mean "to try to invent a perpetual mobile of the second kind".

The paper is organized as follows. The next section is devoted to the xenium engine. In Sec. 3 we discuss the standard proof of the Clausius inequality. Under close examination, it is based on implicit assumptions which are wrong for the

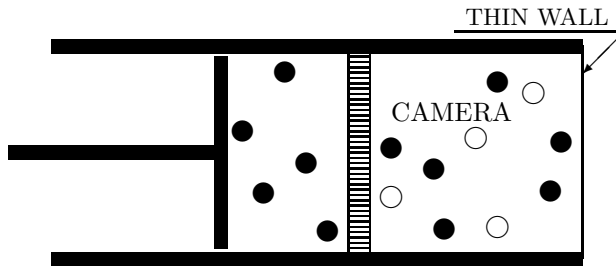


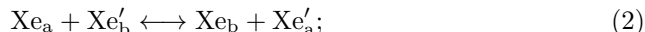
FIGURE 1. The xenium engine

xenium engine. A reformulation of the Clausius inequality is proposed in Sec. 4. The definition of entropy is discussed in Sec. 5. The Szilard engine is considered in Sec. 6. In Sec. 7 the possibility of experimental test of the Clausius inequality is discussed. The recent progress in feedback control [14] makes such a test sufficiently realistic. The conclusions are summarized in Sec. 8.

## 2. THE XENIUM ENGINE

In this section we consider a kind of heat engine. We call it a *xenium engine*. The working body of the engine is an imaginary gas with specific properties. We call this gas *xenium*, denoted by the symbol  $\text{Xe}$ .

Xenium is an ideal gas. A molecule of xenium is in either of two states, denoted by  $\text{Xe}_a$  and  $\text{Xe}_b$ . The energy levels of the states are the same. A single molecule can never change its state. However, two sufficiently close molecules may exchange their states because of a specific interaction. This may be considered a sort of chemical reaction:



(here  $\text{Xe}$  and  $\text{Xe}'$  denote two different molecules, considered as classical particles). There is a resemblance to the electron self-exchange, but no particle like electron is supposed to be transferred. The total number  $N_a(N_b)$  of the  $\text{Xe}_a(\text{Xe}_b)$  molecules does not change. Thus, xenium is a mixture of two gases to some extent.

The xenium engine is a cylinder with a piston which moves without friction (Fig. 1). The wall opposite to the piston is adiabatic. It is also thin in a sense explained below. The cylinder is divided in two by a semipermeable partition. The  $\text{Xe}_a$  molecules can penetrate it while the  $\text{Xe}_b$  ones can not. The space between the thin wall and the partition, called a *camera*, is filled by a mixture of  $\text{Xe}_a$  and  $\text{Xe}_b$ . Another part of the cylinder is filled with pure  $\text{Xe}_a$ .

Consider first a single engine in contact with a heat bath at temperature  $T$ . The piston moves in a quasistatic (hence reversible) process. The pressure  $P$  on the piston is then equal to the partial pressure of  $\text{Xe}_a$  in the camera. By the Gay-Lussac law,

$$P = N_a k_B T / V,$$

where  $k_B$  is the Boltzmann constant and  $V$  is the volume between the piston and the thin wall. It is convenient to consider the dimensionless volume  $v = V/V_0$ , where  $V_0$  is the (constant) volume of the camera. Work in the process is

$$\delta W = -PdV = -N_a k_B T dv.$$

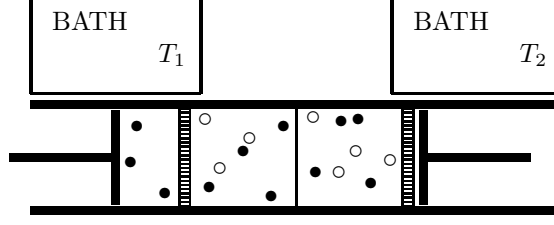
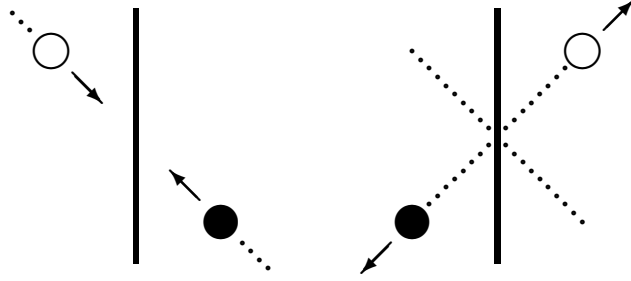


FIGURE 2. Pair of xenium engines

FIGURE 3. Interaction through the thin wall. A molecule of  $\text{Xe}_a$  (black ball) turns to  $\text{Xe}_b$  (white ball), and vice versa.

The internal energy of the gas does not depend on volume, hence  $\delta Q = -\delta W$  and

$$\frac{\delta Q}{T} = N_a k_B d \ln v. \quad (3)$$

Now consider two xenium engines connected as in Fig. 2. Each engine is in contact with a particular heat bath. The adiabatic wall separating the engines is so thin that xenium molecules in one camera may interact with molecules in another camera. The interaction looks similar to a diffusion (Fig. 3).

To distinguish variables related to different engines we use subscripts ‘1’ and ‘2’. While the total number of  $\text{Xe}_a$  molecules  $N_{a,1} + N_{a,2}$  remains a constant, the summands became functions of two variables  $v_1$  and  $v_2$ . It is not difficult to find this functions explicitly. Denote by  $z$  the quotient of the concentrations of  $\text{Xe}_a$  and  $\text{Xe}_b$  in the camera:

$$z = \frac{[\text{Xe}_a]}{[\text{Xe}_b]} = \frac{N_a}{v N_b}.$$

Then

$$N_a = \frac{N}{1 + v^{-1} z^{-1}},$$

where  $N = N_a + N_b$  is a constant for each engine.

By symmetry, the equilibrium constant of the reaction (2) is one. Thus, it comes to equilibrium when  $z_1 = z_2$ . The process is quasistatic, hence this equality holds all the time. In the simplest case  $N_1 = N_2 = N_{a,1} + N_{a,2} = N$  we have

$$z_1 = z_2 = \frac{1}{\sqrt{v_1 v_2}}, \quad N_{a,1} = \frac{N \sqrt{v_1}}{\sqrt{v_1} + \sqrt{v_2}}, \quad \frac{\delta Q_1}{T_1} = \frac{2k_B N}{\sqrt{v_1} + \sqrt{v_2}} d\sqrt{v_1}.$$

Clearly,  $\delta Q_1/T_1$  is *not* an exact differential. It is not difficult to invent a cyclic process such that the Clausius inequality will be violated for this engine. However, the sum

$$\frac{\delta Q_1}{T_1} + \frac{\delta Q_2}{T_2} = 2k_B N d \ln(\sqrt{v_1} + \sqrt{v_2})$$

is an exact differential! (A direct computation shows that it is true for a general choice of parameters as well).

The conclusions are as follows. The xenium engine undergoes a reversible process, but the fraction  $\delta Q/T$  is not an exact differential. Thus, the Clausius inequality can be violated. One can see that a pair of xenium engines is working as a single Carnot engine. Thus, no contradiction to the second law may appear. We have to admit that the Clausius inequality is not a consequence of the second law alone, without extra assumptions. If it were, a situation when the second law is valid while the inequality is not, would not be possible.

In fact, the behavior of the xenium engine can be described by standard thermodynamics. In this case we have, however, to treat the engine as if it were not a closed system. The entropy  $S$  of the xenium engine is the sum

$$S = S_a + S_b,$$

where  $S_a(S_b)$  is the entropy of  $\text{Xe}_a(\text{Xe}_b)$ . The temperature is fixed, and xenium is an ideal gas, hence

$$S_a = -N_a k_B \ln[\text{Xe}_a], \quad S_b = -N_b k_B \ln[\text{Xe}_b].$$

Taking into account the constrain  $dN_b = -dN_a$ , we have the following formula for the change in the entropy of the engine

$$dS = N_a k_B d \ln v - k_B \ln z dN_a.$$

Thus,

$$dS = \frac{\delta Q}{T} - k_B \ln z dN_a.$$

The “wrong” term appears due to the change of composition, which, from a purely formal point of view, is not possible for a closed system.

The fundamental equation for the xenium engine is

$$dU = T dS - P dV + (\mu_a - \mu_b) dN_a,$$

where  $\mu_a(\mu_b)$  is the chemical potential of  $\text{Xe}_a(\text{Xe}_b)$ ; note that for ideal gas

$$\mu_a - \mu_b = k_B T \ln[\text{Xe}_a] - k_B T \ln[\text{Xe}_b] = k_B T \ln z.$$

The third term on the right hand side is, of course, the chemical work.

Consider now a pair of xenium engines. We have the constrain  $dN_{a,2} = -dN_{a,1}$ , hence the change in the total entropy due to the interaction through the thin wall is given by

$$dS_1 + dS_2 = -k_B (\ln z_1 - \ln z_2) dN_{a,1}.$$

The equilibrium condition is then  $z_1 = z_2$ , as expected. One can see that the “wrong” terms in the total entropy are canceled:

$$dS_1 + dS_2 = \frac{\delta Q_1}{T_1} + \frac{\delta Q_2}{T_2}.$$

Thus, it is clear that the origin of the Clausius inequality violation is the exchange of entropy between two engines due to the “chemical” interaction.

The above entropy calculation leaves one unpleasant question. As is well known, the very definition of entropy in classical thermodynamics is based on the Clausius inequality. As the inequality is not valid, one might ask what exactly the entropy of a system is and whether it can be defined properly at all. This reasonable question is answered in Sec. 5.

### 3. ENVIRONMENT DEPENDENT PROCESS

In the following two sections we are going to scrutinize the foundations of the Clausius inequality. First of all, we need to set up the terminology. A system in this paper is a closed thermodynamic system in thermal equilibrium. The latter means that whenever the system is in contact with a heat bath, it is in thermal equilibrium with it. Thus, the temperature  $T$  in (1) is the temperature of the system itself as well. Heat  $Q$  is the energy transferred to the system from a heat bath and work  $W$  is the energy transferred to the system from an external agent. By the first law of thermodynamics,

$$\Delta U = Q + W,$$

where  $U$  is the internal energy of the system. For example,  $W = -Q$  in a cyclic process. As usual, we call a process reversible if it is possible to restore the system as well as the environment to the initial state.

The question under consideration is if the Clausius inequality (1) is true for any cyclic process. Many different proofs of this inequality are known. We have no need to discuss any of these proofs in detail. They all have essentially the same gap. It is sufficient to consider a system undergoing an isothermal cyclic process in contact with a single heat bath. Suppose that the Clausius inequality is violated, i.e.  $Q > 0$ . Thus, heat is taken from the bath and, by the first law, converted to work. Is it in contradiction to the second law? Not yet.

The Kelvin-Planck formulation of the second law states that *it is not possible to take heat from a single heat bath and convert it to work in a cyclic process*. The point is, what is a “cyclic process” in this statement. Naturally, *any* system is supposed to undergo a cycle, with the exception of the bath. Feynman [3] put it as “a process whose **only** net result is to take heat from a reservoir and convert it to work is impossible”. All the proofs of the Clausius inequality are based on the implicit assumption that nothing is changed in the environment. Of course, the inequality  $Q > 0$  is not possible in this case. The problem is whether this assumption can be justified or not.

To make the argument clear, let us take the following definition. Call an *adiabatic* process *environment independent* if a system may undergo it causing no change in the environment. An *isothermal* process is environment independent if a system may undergo it in contact with a single heat bath in such a way that the couple system+bath undergoes an environment independent adiabatic process as a whole. A general process is environment independent if it can be replaced by a combination of isothermal and adiabatic processes of this kind.

We call a process *environment dependent* if it is not environment independent. It is clear from the definition that if a system undergoes an isothermal process with this property then either the environment is changed or heat is given (taken) to the bath from (by) the environment. (It implies that no system can undergo a process of this kind when there is this system and a heat bath and nothing else. This is why the author chose the word “dependent”). The inequality  $Q > 0$  for an isothermal

cycle does not contradict the second law in this case, for the best of our knowledge at least.

The xenium engine considered in Sec. 2 is an example of a system undergoing an isothermal environment dependent process. Indeed, if nothing is changed in the environment, then  $dN_a = 0$ , because the total number of  $\text{Xe}_a$  molecules is unchanged. Thus, any process with  $dN_a \neq 0$  is environment dependent. For such a process the Clausius inequality is not proved, and is not valid.

#### 4. THE WEAK CLAUSIUS INEQUALITY

We have seen that the Clausius inequality cannot be proved for an environment dependent cycle. Nevertheless, it can be proved for an environment independent cycle by a slight variation of the standard method [7]. Consider a system undergoing an environment independent cyclic process. We can replace it by a combination of environment independent isothermal and adiabatic processes. Thus, all the environment remains unchanged, except for a heat bath or several bathes. It is important that any bath in this process exchange heat with the system only. Denote by  $Q_j$  the heat taken by the system from the bath at temperature  $T_j$ . Then, the heat taken by this bath is  $-Q_j$ .

Let us introduce one more heat bath at temperature  $T_0$ . With the help of the Carnot engine (= environment independent reversible cyclic device) we can restore every bath, save this one, to its initial state by giving it heat  $Q_j$  at the expense of heat taken from the exceptional bath at temperature  $T_0$ . The net result of the process will be to take heat  $Q_0$  from this bath and convert it to work. By the second law,  $Q_0 \leq 0$ . Taking into account the properties of the Carnot engine, we have the equality

$$\sum_j \frac{Q_j}{T_j} = \frac{Q_0}{T_0},$$

and the Clausius inequality follows. Going to the limit, we can replace the sum by the integral and write it in the common form (1).

The argument fails for a system undergoing an environment dependent process because of the change in other systems which should be taken into account. Consider a number of systems undergoing a cyclic process together. Following the classical scheme, we connect all the systems to the same bath at temperature  $T_0$  through the Carnot engines (Fig. 4). We suppose that any system involved in the process is taken into account, hence the rest of the environment remains unchanged. Then, by the second law,  $Q_0 \leq 0$ , where  $Q_0$  is the heat taken from the bath. However, the quotient  $Q_0/T_0$  is now equal not to a single integral but to the sum of integrals taken over all the systems. Thus, the inequality we have is the following

$$\sum_i \oint \frac{\delta Q_i}{T_i} \leq 0, \quad (5)$$

where the subscript  $i$  is related to  $i$ -th system. Call this a *weak Clausius inequality*.

In the case of a reversible process (5) turns to an equality. By the common argument, there exists a function  $S$  such that

$$dS = \sum_i \frac{\delta Q_i}{T_i}.$$

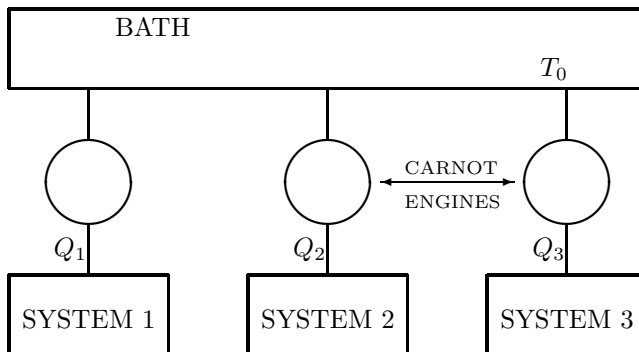


FIGURE 4. Three systems connected to a heat bath.

We have seen an example in Sec. 2. Obviously,  $S$  is the total entropy. The definition of the entropy of an individual system is considered in the next section.

## 5. ENTROPY

In classical thermodynamics the definition of entropy is grounded on the Clausius inequality. What happens to entropy if the inequality fails? The answer to this question become obvious as soon as we realize that any process in the classical setting is implicitly assumed to be environment independent. All we have to do is to make this assumption explicit. Thus, the entropy of a system should be defined by the familiar formula

$$S_B - S_A = \int_A^B \frac{\delta Q}{T},$$

where  $S_A(S_B)$  denotes the entropy of the system in the state  $A(B)$ , and the integral is taken over an arbitrary *environment independent* reversible process  $A \rightarrow B$ . From the previous section we know that for an environment independent cycle the Clausius inequality is valid, hence the definition is sound. Of course, we have to assume that any two states may be connected by an environment independent process (which is not obvious sometimes).

By definition, the change in entropy in a reversible environment independent process is  $dS = \delta Q/T$ , but for a general reversible process this equality may be wrong. Denote the difference by

$$\delta \mathfrak{S} = dS - \frac{\delta Q}{T}. \quad (6)$$

Call  $\delta \mathfrak{S}$  the (infinitesimal) *adiabatic entropy*, taken by the system. Note that by the weak Clausius inequality

$$\sum_i \mathfrak{S}_i = 0,$$

where the sum is taken over all the systems involved in the process. Thus, adiabatic entropy is a form of entropy transfer, by the same way as work is a form of energy transfer.

If the process is neither environment independent, nor reversible, then the equality (6) turns to an inequality. Taking the integral over a cycle, we get the following



generalization of the Clausius inequality

$$\oint \frac{\delta Q}{T} \leq -\mathfrak{S}. \quad (7)$$

(Here is a subtle point. To define adiabatic entropy taken in an irreversible process we have to assume that all the environment undergoes a reversible process. The adiabatic entropy taken by a system is then the adiabatic entropy given by the environment).

In Sec. 2 the entropy of the xenium engine was calculated by standard thermodynamic rules. Here we present a formal proof that it is the “right” entropy. (Apparently, there is no real need of such a proof. It is given by way of illustration of the argument). First of all we have to make sure that any process under consideration is environment independent. For this reason we consider a *single* xenium engine attached to a heat bath. This system has two parameters:  $v$  and  $N_a$ . The simplest process is the piston moving. In this case we have, by (3),

$$dS = N_a k_B d \ln v, \quad N_a = \text{const.}$$

Then, we run into an obstacle: in our model the parameter  $N_a$  is a constant in any environment independent process, so we cannot measure the entropy difference between the states with different numbers  $N_a$ . To circumvent this obstacle we have to make the model a bit more realistic. Let us suppose that there exists a catalyst which makes xenium molecules undergo spontaneous transitions  $\text{Xe}_a \leftrightarrow \text{Xe}_b$ . Consider the following process. At the beginning  $N_a = N/2$  and  $v = 1$ . We add catalyst into the camera and move the piston out. Due to the catalyst, we have  $z = 1$  all the time, hence  $N_a = N/(1 + v^{-1})$ . The entropy change, of course, is given by the same formula  $dS = N_a k_B d \ln v$ .

This is enough to find the entropy in the case  $N_a \leq N/2$ , but the direct computation is a bit ugly. (To access states with  $N_a > N/2$  we need a more complicated process). Instead, one may notice that for the both processes we have the equality

$$dS = N_a k_B d \ln v - k_B \ln z dN_a;$$

the second term on the right hand side vanishes either because of  $dN_a = 0$ , or because of  $\ln z = 0$ . But there are exact differentials on the both sides, hence the equality remains true in general. As one might expect, the adiabatic entropy taken by the xenium engine is

$$\delta \mathfrak{S} = \frac{\partial S}{\partial N_a} dN_a,$$

that is, it is the entropy gained due to the change of composition.

The work done on a system in a reversible isothermal cyclic process can be found from (7):

$$W = T \mathfrak{S}.$$

So, the total work performed by a pair of xenium engines is zero if  $T_1 = T_2$ . In this case the process can be described by the diagram

$$\text{heat} \longrightarrow \text{work} + \text{adiabatic entropy} \longrightarrow \text{heat}$$

One of the engines takes heat from a bath, converts it to work and gives adiabatic entropy to another engine. The latter converts work back to heat and gives heat to another bath.

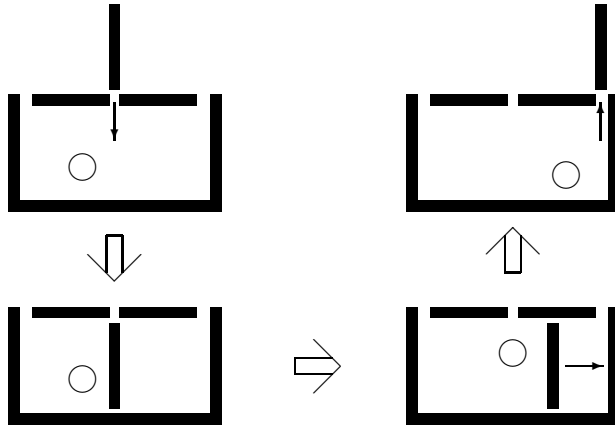


FIGURE 5. The Szilard engine

## 6. THE SZILARD ENGINE

There is a branch of thermodynamics where the Clausius inequality violation cannot be ignored. It is thermodynamics of a feedback controlled system or, to be more precise, a branch of thermodynamics which deals with problems commonly related to the famous Maxwell's demon [9]. A common property of a feedback controlled system is the ability to perform work in an isothermal cycle. Such a behavior formally contradicts both the Clausius inequality and the second law. There exists a well known explanation of why the contradiction to the second law is not real. But, under close examination, the Clausius inequality is broken indeed.

Here we consider the Szilard engine which is, so to say, the prototype of a feedback controlled system. This imaginary device was invented by L. Szilard in 1929 [11]. It consists of a box with a single particle. The box is provided with a thin piston which can be inserted to or removed from it as necessary (Fig 5). The engine undergoes an isothermal cyclic process in contact with a heat bath at temperature  $T$ . At the beginning, the piston is out of the box. As the first step of the process it is inserted into the box at the middle, dividing it into two parts of equal volume. The particle gets trapped in one of the halves. After that, the piston moves into the empty half until it reaches the wall. The piston is then removed and the cycle is complete.

It is supposed that the particle thermalizes in any collision with the walls. It is also supposed that the single-particle gas expands reversibly. Under this assumptions the Gay-Lussac law is valid:  $P = k_B T / V$ . Thus, the work performed in the cycle on the system is

$$W = - \int_{V/2}^V P dV = -k_B T \ln 2.$$

The problem considered in the literature is why the engine cannot violate the second law. We consider a different problem: why the engine *can* violate the Clausius inequality. There exists a quite extensive literature on the Szilard engine [1, 2, 9, 10], so we have no need to discuss numerous technicalities related to this complicated subject.

It is well known that peculiar thermodynamic properties of the Szilard engine are related to the fact that it cannot work by itself. It needs a *controller*. This is a device which makes a *measurement* to find out in which half of the box the particle gets trapped and drives the engine. The Szilard engine cannot work in the absence of a controller. Once a controller is taken into account properly, the paradox with the apparent violation of the second law dissolves. For the best of author's knowledge, this was first noticed by C.H.Bennett [1]; now it is a commonplace.

The presence of a controller, however, does not alter the fact that the Szilard engine performs work in an isothermal cycle. Thus, while the second law is beyond doubt, a paradox remains. Actually, it has been discussed in the literature [10, 5, 9], but the discussion was mostly limited to statistical mechanics. The Clausius inequality was virtually ignored, and the problem was interpreted as a misbehavior of the Boltzmann-Gibbs entropy. (Not the Clausius entropy. For a realistic system it is essentially the same entropy, but an approach makes a difference).

Apparently, the only work where the Clausius, i.e. thermodynamic entropy of the Szilard engine is considered, is the paper of Ishioka and Fuchikami [5]. In their paper, however, the Clausius inequality is not mentioned<sup>1</sup>. According to [5], the Clausius entropy of the engine decreases by  $k_B \ln 2$  when the piston is inserted into the box<sup>2</sup>. This effect cannot be explained in terms of the usual classical thermodynamics, for a very simple reason. The insertion is not a process, it is in fact two different processes, with the same initial state but with different final states. The point is, the choice between the processes is random. A “macroscopic randomness” [10] of this kind is beyond the scope of conventional classical thermodynamics, which is based entirely on the determinism.

The author's suggestion is to extend slightly the framework of thermodynamics by introducing the following variant of the Clausius equality

$$\oint_{\mathcal{A}} \frac{\delta Q}{T} = k_B \ln \frac{P(\mathcal{A}^{-1})}{P(\mathcal{A})}, \quad (8)$$

where  $\mathcal{A}$  is a cyclic process, reversible in a sense. Here  $P(\mathcal{A})$  denotes the probability of  $\mathcal{A}$ , i.e. the probability of success in an attempt to make a system undergo this process. The equality is certainly true for the Szilard engine and the generalizations (like a “skewed” engine [9]), but its range of validity in general is not certain. Under appropriate assumptions it can be proved by means of statistical mechanics, by phase space calculus or following the lines of [13], but the author would like to focus attention on the thermodynamic side of the picture. In classical thermodynamics, given the state of the art, the equality (8) may only be regarded as a plausible conjecture. The author prefers to consider it a possible formulation of the so-called Landauer's principle [8].

A violation of the Clausius inequality by the Szilard engine can, and should, be explained at two different levels. At the low level, it is a consequence of “macroscopic randomness”. Depending on the location of the particle after the insertion, the engine undergoes one of two processes with equal probability,  $P(\mathcal{A}_1) = P(\mathcal{A}_2) =$

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<sup>1</sup>The same authors did mention the Clausius inequality in the preprint [4], and admitted that it is *valid*, though not with confidence. (Which may be considered a forgivable mistake). However, in a subsequent paper [5], which provides a much more detailed account of the same subject, there is no single word about the inequality. This makes the author quite sure that the provocative question if the inequality is true for the Szilard engine was avoided by intention.

<sup>2</sup>The same is true for the Boltzmann-Gibbs entropy [10]

1/2. The reverse processes are both deterministic,  $P(\mathcal{A}_1^{-1}) = P(\mathcal{A}_2^{-1}) = 1$ . By (8),  $Q = k_B T \ln 2$  in any case.

At the high level, it is a consequence of environment dependence. (Note that, strictly speaking, we cannot apply this concept to a process which is not deterministic. To circumvent this obstacle we have to consider the “probabilistic” part of the process, from the insertion of the piston to the memory erasure in a controller, as a whole, without separating it into stages.) The very fact that the engine does not work properly in the absence of a controller implies that it undergoes an environment dependent process. For this reason, the fact that the engine performs work in an isothermal cycle does not contradict anything. Following the same protocol as in the case of the xenium engine, we can attach the Szilard engine to a heat bath and the controller to another heat bath. The standard analysis [1, 2, 10, 5] then shows that the entropy of the former bath decreases by  $k_B \ln 2$  per cycle while the entropy of the latter one increases by the same amount. There is no heat exchange between the engine and the controller, hence there is adiabatic entropy transfer between these systems.

## 7. TESTING THE VIOLATION OF THE CLAUSIUS INEQUALITY

In this section we discuss briefly the possibility of experimental detection of the violation of the Clausius inequality. For obvious reasons we may restrict ourselves by isothermal processes. For an isothermal cycle the Clausius inequality is equivalent to the work inequality  $W \geq 0$ . A violation of the latter inequality for the *total* work is forbidden by the second law of thermodynamics. However, we may consider a process which involves several interacting systems. If  $W_i$  denotes the work done on  $i$ -th system, then, by the second law,

$$\sum_i W_i \geq 0.$$

On the other hand, by the Clausius inequality,  $W_i \geq 0$  for *each* system. The latter is a more strong statement than the former. Due to this difference, a violation of the Clausius inequality under appropriate conditions does not imply an opportunity for building a perpetual mobile of the second kind.

A promising approach to test the Clausius inequality is to perform an experiment with feedback control. All we need here are some basic principles. We follow the convenient notation of [12]. There are three thermodynamic systems taking part in the process: a controlled system S, a memory (or controller) M, and a heat bath B. Let  $W^S$  be the work done on S and  $W^M$  be the work done on M. For an isothermal cyclic process we have two inequalities:

$$W^M \geq k_B T I, W^S \geq -k_B T I,$$

where  $T$  is the temperature of B and  $I \geq 0$  is the mutual information. The quantity  $I$  may be interpreted as the amount of information about S, obtained during the cycle. It follows that  $W^S + W^M \geq 0$ , in agreement with the second law.

In the process considered in [12] no violation of the Clausius inequality is possible, because the cycle is not actually complete (the state of S is changed). To make the process a proper cycle we must extend it. Let S be initially in thermodynamic equilibrium in contact with B. Then it is detached from B and the process goes on as in [12]. Finally, S is attached to B again. We have then an isothermal cyclic

process. If  $I > 0$  then the possibility of the following inequality

$$-k_B T I \leq W^S < 0$$

is not excluded by any known principle. This possibility, if realized, would imply a violation of the Clausius inequality by S. This consideration is in fact very general and independent of the details of the process. To test the Clausius inequality we have simply to measure the work  $W^S$ .

A successful feedback control experiment has been recently performed [14]. In the experiment, a small particle is rotated against the applied moment of a force, at the expense of heat taken from the environment. Though this experiment was not designed to test the Clausius inequality, the author is at the opinion that the possibility of a violation of the inequality under similar conditions ought to be discussed. The violation, of course, is by no means obvious. One of the major issues is the influence of the measurement on the process. (The particle is illuminated to find out its position. The light is absorbed partially by the particle and the medium, hence the process is not exactly isothermal).

An interesting question is if the Clausius inequality can be violated on macroscopic scale. Quantitatively, the question is about the possibility of “large scale” adiabatic entropy transfer  $\mathfrak{S}/k_B \gg 1$ . At present, the author does not know how such an experiment can be devised. On the other hand, he does not know about any fundamental obstacle either.

## 8. CONCLUSIONS

In this paper the limitations of the Clausius inequality are discussed. It is shown that in the general case the inequality is not a consequence of first principles of thermodynamics. Thus, the Clausius inequality violation is not forbidden, for the best of our knowledge.

Such a violation, if it is possible, can be explained in terms of thermodynamics by adiabatic entropy transfer between two systems. In this hypothetical process entropy is transferred from one system to another while there is neither energy transfer nor exchange of particles. This transfer takes place in some thought experiments, including the well known Szilard engine.

The possibility of the Clausius inequality violation in nature is not excluded by any known principle of physics. So, it would not be unreasonable to test it in a laboratory. The recent progress in feedback control experiments makes such a test realistic enough.

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